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Elastica hypoarealis

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Abstract. We examine the equilibria of a rigid loop in the plane, characterized by an energy functional quadratic in the curvature, subject to the constraints of fixed length and fixed enclosed area. Whereas the only non self-intersecting equilibrium corresponding to the fixed length constraint is the circle, the area constraint gives rise to distinct equilibria labeled by an integer. These configurations exhibit self-intersections and bifurcations as the area is reduced. In addition, not only can the Euler-Lagrange equation be integrated to provide a quadrature for the curvature but the embedding itself can be expressed as a local function of the curvature. Perturbations connecting equilibria are shown to satisfy a first order ODE which is readily solved. Analytical expressions for the energy as a function of the area are obtained in the limiting regimes.

PACS. 46.70.Hg Membranes, rods and strings – 87.16.Dg Membranes, bilayers, and vesicles

1 Introduction

Consider a closed planar loop with an elastic energy density proportional to the square of its curvature. If we fix the length of this loop the equilibria are well known [1]. There is a unique configuration for each value of the winding number m associated with the rotation of its normal: the obvious ones consisting of the circle and its multiple coverings, as well as the figure of eight ($m = 0$). Suppose, in addition, that we fix the area enclosed by the loop to lie below its circular value (hence the term *hypoarealis*). What are then its equilibria? The characterization of these equilibria is a subtle issue with several surprises: while it has been known for a long time that the Euler-Lagrange equation is integrable to provide a quadrature for the curvature [2], we find that the position vector $\mathbf{X}$ describing the embedding of the curve in the plane is also a local function of the curvature; the closure of the loop quantizes the energy parameter associated with the quadrature and determines the curvature.

In particular, as the enclosed area is reduced, how does the circular equilibrium deform? One finds that there is an infinite number of configurations with an n-fold symmetry, $n = 2, 3, \ldots$ For each n there is a critical area below which the configuration self-intersects; there is also a limiting area below which the equilibrium is impossible. In addition, there are configurations which bifurcate from a complicated limiting shape which cannot be deformed into a circle. A detailed analysis is provided in [3]. The problem we address is relevant to the study of polymers and membranes as well as solitons.

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2 Euler-Lagrange equation and scaling

The energy functional that implements the constraints is

$$F_c = \int \mathrm{d}\ell \left(k^2 + \mu + \frac{1}{2}\sigma \mathbf{n}\cdot\mathbf{X} \right) . \quad (1)$$

Here k is the geodesic curvature of the loop, ℓ is the arclength, $\mathbf{n}$ is the normal to the loop and μ, σ are Lagrange multipliers. In terms of the normal component ϵ of a small variation $\delta\mathbf{X}$, the variation of the curvature is $\delta_\epsilon k = -\epsilon'' - k^2\epsilon$, which leads to the Euler-Lagrange equation

$$2k'' + k^3 - \mu k - \sigma = 0 ; \quad (2)$$

the prime indicates a derivation with respect to arclength. This equation may be recast as $k'' = -\mathrm{d}V(k)/\mathrm{d}k$, where $V(k) = k^4/8 - \mu k^2/4 - \sigma k/2$. Thus, with ℓ identified as time, its solutions can be identified with the motion of a particle (with position k) in a quartic potential. The energy E of this particle (not to be confused with the elastic energy) is conserved along the loop; $k(\ell)$ is determined by the elliptic integral,

$$\ell = \int \mathrm{d}k[2(E - V(k))]^{-1/2}. \quad (3)$$

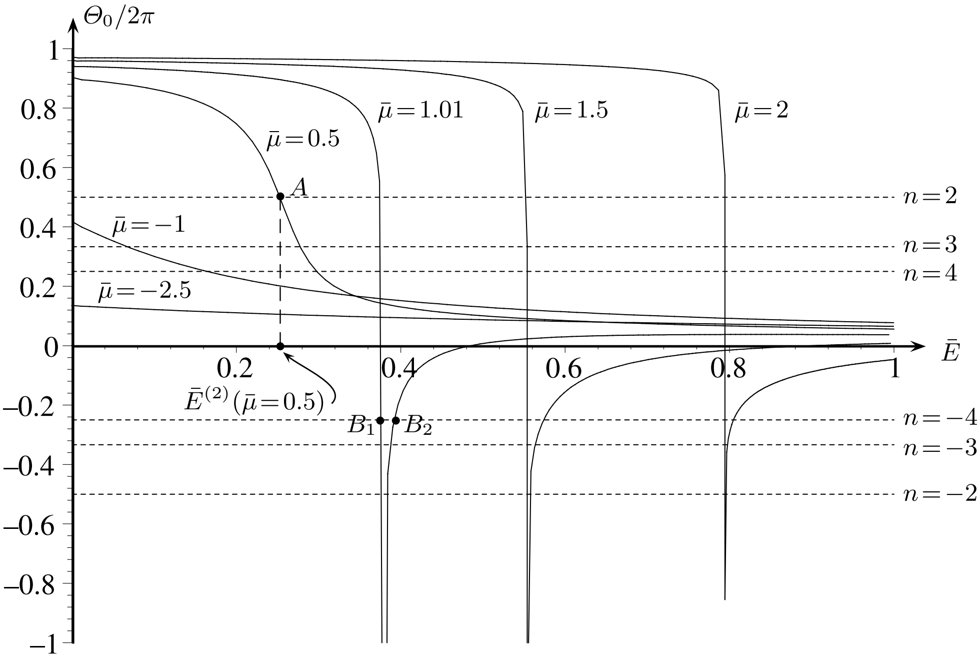

Fig. 1. The angle Θ_0, as defined by equation (4), *vs.* $\bar{E} \equiv (E - V(K_+))/\mu_0^2$, for various values of $\bar{\mu} \equiv \mu/\mu_0$. Also shown, with horizontal dashed lines, are some of the values of Θ_0 that give rise to closed configurations. The corresponding values of $\bar{E}$ can be read off as in the case $\bar{\mu}=0.5$, $n=2$ shown. Notice how the $\bar{\mu} = -1$, -2.5 curves miss the $n = 2$, $n = 2, 3, 4$ configurations respectively.

One would then expect the determination of the position vector $\mathbf{X}(\ell)$ to involve two further integrations. It is remarkable, as we will discuss below, that no further integrations are necessary.

Scaling may be exploited to reduce the two dimensional parameter space (σ, μ) by one. If $\mathbf{X}$ is a solution then $\lambda\mathbf{X}$ is also a solution if the parameters are rescaled as $(\sigma, \mu) \to (\lambda^{-3}\sigma, \lambda^{-2}\mu)$. Moreover, if $\lambda = -1$, then the inversion $\mathbf{X} \to -\mathbf{X}$ produces a solution with the sign of σ flipped. Thus, we need only to consider $\sigma > 0$. Until further notice we will set $\sigma = 1$.

3 Closure

The particle analogue is only one part of the story. We still need to implement the closure of the loop. For each value of μ this condition will quantize the energy E.

Consider one complete oscillation of the particle in the potential falling from its maximum down to its minimum and back. The corresponding angle Θ_0, by which the normal vector is rotated, is then given by

$$\Theta_0(E, \mu) = 2 \int \mathrm{d}\ell\, k = 2 \int_{k_{\min}}^{k_{\max}} \frac{k\,\mathrm{d}k}{\sqrt{2(E - V)}} \cdot \qquad (4)$$

Closure requires that

$$\Theta_0 = 2\pi m/n\,, \qquad (5)$$

where $m \in \mathbf{Z}^*$ and $n = \pm 2, \pm 3, \cdots$ ($n = \pm 1$ is excluded by the four-vertex theorem). The values $m \neq \pm 1$ give rise to configurations that self-intersect necessarily. The figure possesses an n-fold axis of symmetry with a well defined center. The value n can be identified with the winding number associated with orbits on the phase plane. There are thus two periodicities intrinsic to this problem which are labelled by n and m. We will confine ourselves to $m = 1$.

We need to determine (i) the functional form of $\Theta_0(E, \mu)$; (ii) those values of E, for each fixed value μ, satisfying the quantization condition. This will then fix $k(\ell)$ as determined by equation (3).

3.1 $\Theta_0(\mathrm{E}, \mu)$

We begin with a qualitative description of $\Theta_0(E, \mu)$ (refer to Fig. 1). If E is large, the linear term in the potential is irrelevant and the approximate symmetry $k \to -k$ implies that $\Theta_0 \approx 0$; closure will only be possible for large values of n. On the other hand, when E is small the details at the bottom of the potential become important; we distinguish between the case $\mu < \mu_0 = 3/2^{2/3}$ where the potential possesses a single minimum at $k_+ > 0$ (with energy E_+), and $\mu > \mu_0$ where a second minimum appears, at $k_- < 0$, with a higher energy E_-. In the latter range μ, for values of E lying between E_- and the energy of the central maximum of $V(k)$, the function Θ_0 will have two values, one for each well.

All configurations which correspond to oscillations in the left well self-intersect. While this branch is undoubtedly of mathematical interest, we will ignore it in this paper.

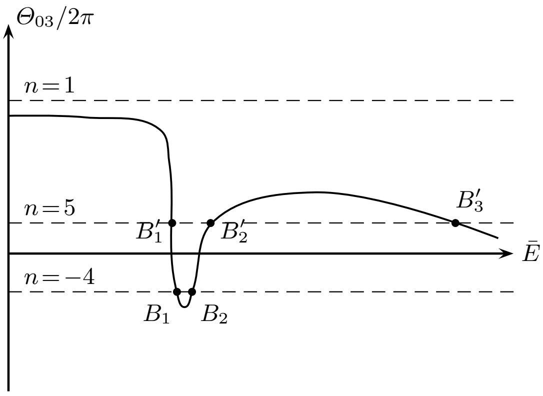

Fig. 2. Bifurcations: as $\bar{\mu}$ is reduced, the minimum of the curve shifts upwards and B_1, B_2 collapse to a point. Further decrease of $\bar{\mu}$ forces B_1', B_2' to collapse to a point. The corresponding configurations cannot be continuously deformed to a circle.

In general, Θ_0 ranges from some positive value at $E = E_+$ to zero as $E \to \infty$. $E = E_+$ is a global maximum of this function. The behavior at intermediate values will depend on μ. For low values of μ, Θ_0 is a monotonically decreasing function of E. There is a critical value μ_c at which a local minimum appears in Θ_0 (note that $\mu_c < \mu_0$), as sketched in Figure 2. As μ is increased further, this minimum progressively deepens. Above μ_0 the minimum transforms into a pole at the energy of the central maximum of $V(k)$ (see Fig. 1).

3.2 The energy spectrum

Now let us return to address the quantization condition, equation (5).

There exists a set of configurations labeled by $n = 2, 3, \cdots$ which are continuously deformable into a circle. A quadratic approximation of the potential can be exploited to obtain

$$\Theta_0(E_+, \mu) = 2\sqrt{2}\pi(3 - \mu/k_+^2)^{-1/2}, \tag{6}$$

where k_+ is the positive root of $k^3 - \mu k - 1 = 0$. In particular, we note that as μ increases from $-\infty$ to ∞, $\Theta_0(E_+, \mu)$ increases monotonically from zero to 2π. This implies that there is a sequence of values $\mu_2 > \mu_3 > \mu_4 \ldots$, such that when $\mu = \mu_n$, the configuration of order n first occurs as a circle. Below μ_n it does not occur; above it it deforms continuously, becoming self-intersecting at some point. Such a sequence is illustrated in Figure 3 for $n = 2$. We note also that a limiting configuration exists for each n consisting of an $(n-1)$-fold covering of the circle decorated with n small circles.

Above μ_c a new set of solutions appears in the spectrum. As μ increases, the minimum of Θ_0 will cross the critical values of equation (5) for increasing values of n. A pair of configurations bifurcate from a single one (see Fig. 2). The number of configurations diverges as the minimum of Θ_0 falls to zero. As Θ_0 passes through zero a new

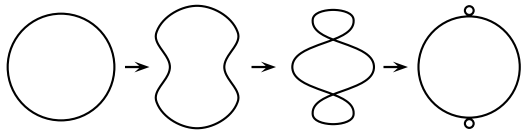

Fig. 3. Evolution of a $n = 2$ configuration under deflation (sketch). The central region in the third configuration, as well as the big circle in the fourth one, contribute negative area.

set of configurations with large negative values of n appear, with all orders appearing as μ increases towards μ_0. These exotic configurations are self-intersecting and are not deformations of a circle.

4 The embedding

Consider two solutions of equation (2), corresponding to (μ, σ), $(\mu+\delta\mu, \sigma+\delta\sigma)$. The normal displacement $\epsilon \equiv \delta\mathbf{X}\cdot\mathbf{n}$ connecting them, satisfies the fourth order ODE

$$DE(\epsilon) = -k\delta\mu - \delta\sigma\,, \tag{7}$$

where $DE(\epsilon) = 2\epsilon'''' + (5k^2 - \mu)\epsilon'' + 10kk'\epsilon' + (12E - \sigma k^4/2 + 6\mu k^2 + 10\sigma k)\epsilon$. The linear operator DE possesses three zero modes corresponding to the three Euclidean motions on the plane: two independent translations $\delta\mathbf{X} = \mathbf{a}$ with $\epsilon = \mathbf{a}\cdot\mathbf{n}$ and a rotation $\delta\mathbf{X} = \omega \times \mathbf{X}$, so that ϵ is proportional the tangential projection of $\mathbf{X}$, $\mathbf{X}\cdot\mathbf{t}$. An infinitesimal dilatation of the loop with ϵ proportional to the projection of $\mathbf{X}$ onto the normal, $\mathbf{X}\cdot\mathbf{n}$, also satisfies equation (7) with appropriate values of $\delta\sigma$ and $\delta\mu$. The normal and the tangential components of $\mathbf{X}$ together describe the configuration completely. They both satisfy equation (7). Are there any other solutions? We find that $DE(k') = 0$ and $DE(k^2 - \mu) = 2\mu\sigma K + 3\sigma^2$. One would not expect two new independent solutions. In fact, one can prove that $\mathbf{X}\cdot\mathbf{n} = \sigma^{-1}(k^2 - \mu)$ and $\mathbf{X}\cdot\mathbf{t} = 2\sigma^{-1}k'$ for all closed configurations centered at $\mathbf{X} = 0$. We note that these identifications become singular in the $\sigma \to 0$ limit, *i.e.*, when the area constraint is relaxed.

An immediate geometrical consequence is that

$$\mathbf{X}^2 - X_0^2 = 4\sigma^{-1}k\,, \tag{8}$$

where $X_0 = \sigma^{-1}(8E + \mu^2)^{1/2}$. This remarkable identity provides a geometrical construction of the curve once k is specified as a function of ℓ. It is worth remarking that an alternative derivation of this result is obtained in the context of the local induction hierarchy by Langer [4]. A further consequence is that ϵ, in fact, satisfies the first order ODE,

$$4k'\epsilon' - 4k''\epsilon = c_2k^2 + c_1k + c_0\,, \tag{9}$$

with the three constants linear in $\delta\sigma$, $\delta\mu$ and δE. This equation is three orders lower in derivatives than (7) and is readily integrated to give

$$\epsilon(k) = \frac{1}{4}k' \int \mathrm{d}k\, \frac{c_2k^2 + c_1k + c_0}{[2(E - V)]^{3/2}}\,. \tag{10}$$

5 Self-intersections

It is desirable to have a condition that excludes self-intersecting configurations. We consider here only self-intersections that can be reached by continuous deformation of non-self-intersecting configurations. We call a "kiss" the point of contact (but not self-crossing) of a curve that bends on itself. At a kiss we have that $\mathbf{X} \cdot \mathbf{n} = \sigma^{-1}(k^2 - \mu) = 0$, from which we find

$$k = -\sqrt{\mu} \tag{11}$$

at a kiss. Therefore, a sufficient condition for non-self-intersection is $\mu < 0$.

6 F vs. A for fixed L

The total elastic energy F, as a function of the enclosed area A, for a fixed perimeter L, may be obtained by interpolating between the two limiting regimes where we possess analytical expressions. We begin by examining the behavior of F for a perturbed circle. We have, in general, at a circular equilibrium, $\mathrm{d}V/\mathrm{d}k = 0$ which implies $R^2\mu + R^3\sigma = 1$, with $R = k^{-1}$. Equations (5, 6) imply $\sigma = 2(n^2 - 1)/R^3$ for a deformation of a circle of order n. It can be shown that, for a fixed length $L = 2\pi R$, $\partial F/\partial A = -\sigma$. Thus

$$F \approx \frac{2\pi}{R} - \frac{2(n^2-1)}{R^3}(\pi R^2 - A)\,. \tag{12}$$

At the limiting geometry with n small circles, F possesses a pole at $A = -\pi R^2/(n-1)$. The residue is determined by the n decorating circles

$$F \approx \frac{4\pi n^2}{R}\frac{1}{1 + (n-1)\frac{A}{\pi R^2}}\,. \tag{13}$$

7 Concluding remarks

We have examined the configuration space for elastica hypoarealis. It is remarkable how this simple geometrical problem offers an enormous richness of configurations and a few surprises along the way. One important issue is the stability of these configurations. It appears that at least the non-self-intersecting configurations are stable, but a full answer will have to wait further work.

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